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ACOUSTICAL IMPULSE RESPONSE USING MAXIMUM-LENGTH
SEQUENCES AND FFT

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ABSTRACT

In acoustics the reverberation time and the reverberation decay function are central concepts. In this paper the problem of estimating the reverberation decay function is treated. A method in which the impulse response is estimated using a binary maximum-length sequence of period $2^n - 1$ as exciting signal, is analysed, developed and applied. The estimate of the impulse response is found by cross-correlating the input and output signal. This cross-correlation is made by using an FFT algorithm of 2^{n+1} samples. The estimate of the impulse is then used to calculate an estimate of the reverberation decay function. The method is very simple to use and gives high signal-to-noise ratios.

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1. INTRODUCTION

In acoustics the reverberation time and the reverberation decay function are central concepts. These notations are essential for concert halls and many other acoustic applications. In some situations the reverberation time needs to be estimated with a high degree of accuracy. This means that the concepts have to be well defined as done by Schroeder in [1]. Furthermore the measurements have to be made with a high degree of accuracy.

In this paper the problem of estimating the reverberation decay function is treated. From this we can easily calculate the reverberation time. A measuring method, suggested by Schroeder in [2], that gives good estimates is analysed and developed in chapter 2 and 3. In chapter 4 a reverberation decay function from a practical measurement using this method is presented and evaluated.

An acoustic indoor channel may with high accuracy be modelled into a linear, time-invariant and causal system. This means that an impulse response, $h(t)$, completely describes the acoustic channel. The reverberation decay function is then, [1], defined by

$$r(t) = \frac{\int_0^{\infty} h^2(\tau) d\tau}{\int_t^{\infty} h^2(\tau) d\tau} \quad (1.1)$$

The reverberation time, T_R , was originally defined by the value t_R satisfying

$$r(t_R) = 10^{-6} \text{ (-60 dB)}. \quad (1.2)$$

In practice, however, this can seldom be used. Today one often uses the slope between -5 dB and -35 dB, i.e.

$$T_R = 2(T_{-35} - T_{-5}) \quad (1.3)$$

where

$$r(T_{-35}) = 10^{-\frac{35}{10}} \quad (1.4)$$

and

$$r(T_{-5}) = 10^{-\frac{5}{10}}. \quad (1.5)$$

In situations of less dynamic range where $r(T_{-35})$ is difficult to estimate one also makes use of

$$T_R = 3(T_{-25} - T_{-5}). \quad (1.6)$$

In room acoustics the slope of the first part of the reverberation decay function has shown to be important for listening conditions. Schroeders method gives, even for small dynamic ranges, a sufficiently accurate estimate of the "early decay", T_{ed} , defined by

$$T_{ed} = 6(T_{-11} - T_{-1}). \quad (1.7)$$

Providing idealized conditions, "diffuse field", the reverberation decay function becomes

$$r(t) = 10^{-\frac{6t}{T_R}} = e^{-t \frac{6 \ln 10}{T_R}} \quad \text{for } t \geq 0. \quad (1.8)$$

Then the different specifications (1.2), (1.3), (1.6) and (1.7) will give the same result. This is due to the fact that these formulas are matched to (1.8).

We will now briefly study the conventional method, illustrated in Fig. 1.1, of estimating the reverberation decay function.

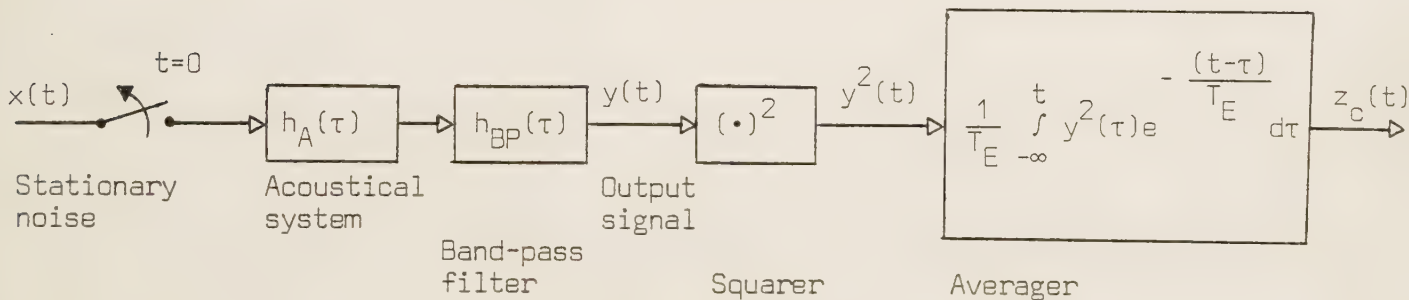


Figure 1.1. Conventional measurement of the reverberation decay function.

The measurement is made using a loudspeaker, a microphone and a filter. The filter, a band-pass filter, is used to define the frequency band for which the reverberation decay function is measured. One uses a zero-mean white noise signal, $x(t)$, that excites the room by means of the loudspeaker. When

the output signal, $y(t)$, from the microphone in the room has become stationary the input signal $x(t)$ is turned off. This gives a non-stationary output $y(t)$. By squaring and averaging this signal, where the averaging is exponentially weighted made for instance by an RC circuit, one obtains a signal denoted by $z_c(t)$. This procedure is then repeated several times and the mean value

$$\bar{z}_c(t) = \frac{1}{N} \sum_{i=1}^N z_c^i(t) \quad (1.9)$$

calculated where t is related to the turn off time for each of the different output signals $\{z_c^i(t)\}_{i=1}^N$. An estimate of the reverberation decay function is defined by

$$\hat{r}_c(t) = \frac{\bar{z}_c(t)}{\bar{z}_c(0)}. \quad (1.10)$$

If N is large enough we have

$$\hat{r}_c(t) \simeq r_c(t) = \frac{E\{z_c(t)\}}{E\{z_c(0)\}} \quad (1.11)$$

where $E\{\cdot\}$ denotes the expected value.

We will therefore use $r_c(t)$ and compare this function with $r(t)$ defined in (1.1). Consider first

$$\sigma_Y^2(t) = E\{y^2(t)\}. \quad (1.12)$$

We find

$$\sigma_Y^2(t) = R_0 \int_t^\infty h^2(\tau) d\tau \quad (1.13)$$

where R_0 is the spectral density of the white noise $x(t)$ and

$$h(\tau) = \int_0^t h_A(\tau) h_{BP}(t-\tau) d\tau. \quad (1.14)$$

Then we have

$$r_c(t) = \frac{1}{T_E \sigma_Y^2(0)} \int_{-\infty}^t \sigma_Y^2(\tau) e^{-\frac{(t-\tau)}{T_E}} d\tau = \frac{1}{T_E} \int_{-\infty}^t r(\tau) e^{-\frac{(t-\tau)}{T_E}} d\tau \quad (1.15)$$

where T_E is the time constant of the averager, i.e. the RC time. In order to further analyse this function we assume that (1.8) is valid. We then have

$$r_c(t) = f(t) \cdot e^{-t \frac{6 \ln 10}{T_R}} \quad (1.16)$$

where

$$f(t) = e^{-t(\frac{1}{T_E} - \frac{6 \ln 10}{T_R})} + \frac{1 - e^{-t(\frac{1}{T_E} - \frac{6 \ln 10}{T_R})}}{1 - \frac{T_E 6 \ln 10}{T_R}} \quad (1.17)$$

In any adequate measurement situation T_E is such that

$$T_E < T_r = \frac{T_R}{6 \ln 10} \quad (1.18)$$

This means that $f(t)$ is an increasing function and

$$1 \leq f(t) < \frac{1}{1 - \frac{T_E 6 \ln 10}{T_R}} \quad (1.19)$$

It is seen that $r_c(t)$ is not exactly equal to $r(t)$. The deviation is however bounded. In dB this bound is

$$D = 10 \log \left(\frac{1}{1 - \frac{T_E 6 \ln 10}{T_R}} \right) \quad (1.20)$$

and for $T_E \ll T_r$ we have

$$D \approx 60 \frac{T_E}{T_R} \quad (1.21)$$

For $T_E \ll T_R/60$ we thus have

$$r_c(t) \approx r(t). \quad (1.22)$$

Let us now study a practical example. There are rooms for which one requires a reverberation time less than or equal to 0.6 seconds. If one takes the con-

ventional method using the valuable Brüel & Kjaer 2131 instrument with the shortest RC-time, 1/32 seconds, the deviation is bounded by

$$D = 10 \log \left(\frac{1}{1 - \frac{6 \ln 10}{32 \cdot 0.6}} \right) \approx 5.5 \text{ dB.} \quad (1.23)$$

This means that one has to be careful when measuring short reverberation times since different RC-times will give different reverberation decay functions and thereby different estimates of the reverberation time.

The definition of the reverberation decay function (1.1) shows that a natural way to estimate $r(t)$ is by first estimating $h(t)$ and then using this result for obtaining an estimate of $r(t)$. There are several ways in which to estimate $h(t)$. A commonly practiced method is to estimate $h(t)$ by means of a short pulse that excites the room. Among others Bodlund, [3], has done this. However, this method gives, using a loudspeaker as exciting device, low signal-to-noise ratios. A more effective method found by Schroeder, [2], uses a binary maximum-length sequence, i.e. a so-called pseudorandom sequence, as exciting signal. This is in literature an often used method for estimating the impulse response. The method uses discrete time signal processing. This means that the analog system has to be transformed into an equivalent discrete time system. This is performed in chapter 2.

There is a filter problem connected with the definition of the reverberation decay function. In acoustics one often measures reverberation over a number of frequency bands, 1/1-octave and 1/3-octave, with centerfrequency from 63 Hz to 8 kHz. This means that a band-pass filter has to be included in the impulse response. One also has to include the amplifiers, the loudspeaker and the microphone. Since these systems only have a standardized amplitude function and not a standardized impulse response they may in some cases cause problems. For example, if one has a bank of 1/3-octave filters and wants to make a 1/1-octave filter measurement, it is done by using three parallel 1/3-octave filters. This filter has an impulse response of an essentially longer duration than that of a normal 1/1-octave filter. This may give deviating reverberation decay functions. In this paper we will however be content to just mention that this problem exists.

2. ESTIMATION OF THE IMPULSE RESPONSE

In this chapter we consider the estimation of the impulse response $h(t)$ using pseudorandom sequences. Although well-known the method has so far not often been used in acoustics. Schroeder, [2], suggests the method for calculating reverberation and Jönsson, [4], uses it in order to estimate the acoustic absorption coefficient.

The method uses a discrete time signal processing technique. This means that the discrete input signal has to be modulated giving an analog input signal and the analog output signal has to be sampled giving a discrete output signal. This is shown in Fig. 2.1a. We note that there is an equivalent discrete system. This system is shown in Fig. 2.1b. Since the analog system has a limited bandwidth, B , we are able to choose a sampling rate such that the analog signal $y_c(t)$ can be reconstructed with arbitrary accuracy by means of the discrete signal $y(k)$. For an ideal bandwidth limited signal one requires that

$$f_s = \frac{1}{T_s} > 2B \quad (2.1)$$

where f_s is the sampling rate and B the bandwidth. For non ideal bandwidth limited signals the sampling rate has to be such that the folding errors become negligible, that is the power or energy outside the bandwidth $f_s/2$ composes a fractional part of the total signal power or energy and is not worth considering.

An analog to digital converter, denoted ADC, is introduced in the measuring system showed in Fig. 2.1a. The quantisation errors of the ADC can be disregarded if the quantisation step, Δ , is such that the power $\Delta^2/12$ is negligible in the measuring system. Let us consider the measuring system in Fig. 2.1a assuming that f_s and Δ are such that the folding and the quantisation errors become negligible.

The input signal $a(k)$ is an infinite binary, +1 and -1, maximum-length sequence with the period

$$N = 2^n - 1 \quad (2.2)$$

The modulator gives the exciting signal $x(t)$ defined by

$$x(t) = \sum_{k=-\infty}^{\infty} a(k) s(t - kT_s) \quad (2.3)$$

where $s(t)$ is a pulse function such that

$$S(f) \cdot H(f) \approx c_s H(f) \quad (2.4)$$

where c_s is a constant. We assume that $c_s=1$. This is equivalent to including c_s in $H(f)$. $S(f)$ and $H(f)$ are the Fourier transforms of $s(t)$ and $h(t)$ respectively. The impulse response $h(t)$ is defined in (2.7). The formula (2.4) means that $x(t)$ is built up of non-overlapping or almost non-overlapping pulses.

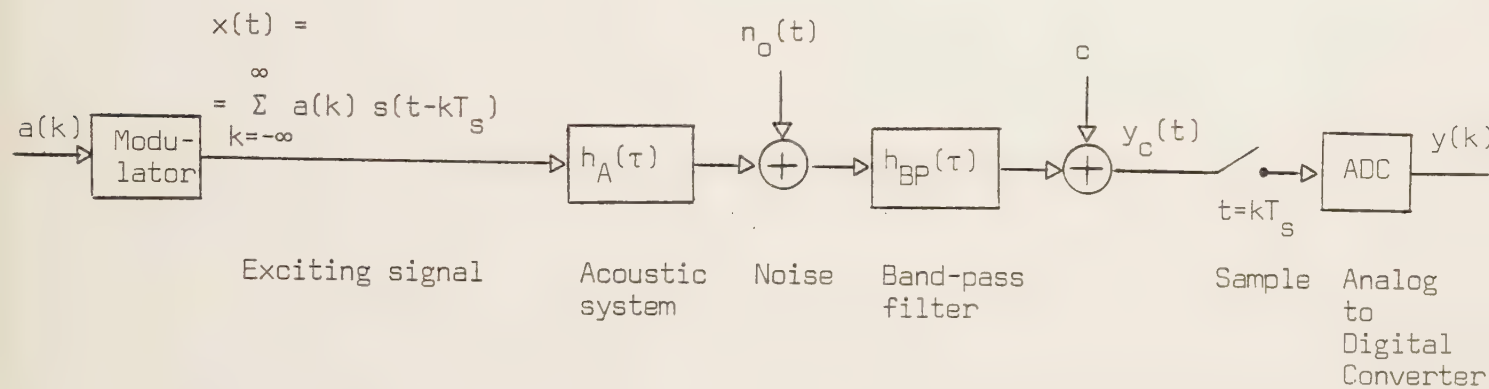


Figure 2.1a. The measuring system.

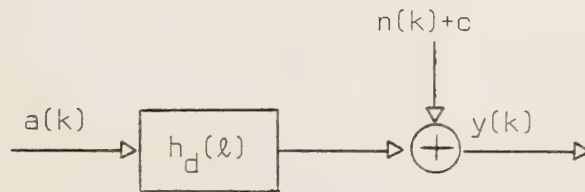


Figure 2.1b. The equivalent discrete time system.

The analog output $y_c(t)$ is

$$y_c(t) = y_a(t) + n_o(t) + c \quad (2.5)$$

where

$$y_a(t) = \int_0^{\infty} h(\tau) x(t-\tau) d\tau, \quad (2.6)$$

$$h(t) = \int_0^t h_A(\tau) h_{BP}(t-\tau) d\tau, \quad (2.7)$$

$$n_c(t) = \int_0^\infty h_{BP}(\tau) n_o(t-\tau) d\tau, \quad (2.8)$$

$n_o(t)$ is zero-mean white noise and the constant c represents the total dc offset of the measuring system including the input amplifier of the analog to digital converter. The function $h_A(\tau)$ is the impulse response of the acoustical system. The sampling rate is

$$f_s = \frac{1}{T_s}. \quad (2.9)$$

The modulator is clocked by this frequency (2.3). The variance of the noise signal $n_c(t)$ is

$$\sigma_n^2 = E\{n_c^2(t)\} = R_o \int_{-\infty}^{\infty} |H_{BP}(f)|^2 df = R_o \cdot 2B \quad (2.10)$$

where $H_{BP}(f)$ is the analog Fourier transform of the impulse response $h_{BP}(t)$ and B the noise bandwidth of the band-pass filter assuming that $|H_{BP}(f)| \approx 1$ in the pass-band. The centerfrequency, the geometric mean, of the band-pass filter is denoted by f_o and for a 1/1-octave filter we have the bandwidth

$$B = f_o / \sqrt{2}. \quad (2.11)$$

The signal-to-noise ratio after the band-pass filter is defined by

$$SNR_{BP} = \frac{1}{NT_s \sigma_n^2} \int_0^{NT_s} y_a^2(\tau) d\tau. \quad (2.12)$$

The assumptions made above give

$$h_d(k) \approx h(kT_s) \quad (2.13)$$

and

$$n(k) = n_c(kT_s). \quad (2.14)$$

The sampling rate, f_s , is when using 1/1-octave filters chosen to be

$$f_s = 4f_0 = 4\sqrt{2} \cdot B. \quad (2.15)$$

The discrete output $y(k)$ is given by

$$y(k) = \sum_{\ell=0}^{\infty} h_d(\ell) a(k-\ell) + n(k) + c. \quad (2.16)$$

The noise $n(k)$ is a zero-mean in wide sense stationary process with the variance σ_n^2 . The pulse response $h_d(k)$ is estimated by cross-correlating the input and output signal, i.e. $a(k)$ and $y(k)$. One then uses the fact that the periodic correlation function of the maximum-length sequence $\{a(k)\}^\infty$ defined by

$$r_{aa}(\ell) = \frac{1}{N} \sum_{k=1}^N a(k) a(k-\ell) \quad (2.17)$$

is given by

$$r_{aa}(\ell) = \frac{N+1}{N} \sum_{j=-\infty}^{\infty} \partial(\ell+jN) - \frac{1}{N} \quad (2.18)$$

where

$$\partial(\ell) = \begin{cases} 1 & \text{for } \ell = 0 \\ 0 & \text{for } \ell \neq 0 \end{cases} \quad (2.19)$$

In some situations it is more suitable to cross-correlate the output $y(k)$ and $b(k)$, where $b(k)$ is given by

$$b(k) = a(k) - \frac{1}{N} \quad (2.20)$$

since the sequence $b(k)$ has no dc component, i.e.

$$\frac{1}{N} \sum_{k=1}^N b(k) = \frac{1}{N} \sum_{k=1}^N \left(a(k) - \frac{1}{N} \right) = 0 \quad (2.21)$$

We estimate $h_d(i)$ from an observation of $y(k)$ over M number of periods, each containing N number of samples, using $z_a(i)$ defined by

$$z_a(i) = \frac{1}{MN} \sum_{k=1}^{MN} y(k) a(k-i). \quad (2.22)$$

We find by using (2.18) and (2.21) that

$$z_a(i) = \frac{(N+1)}{N} h_{df}(i) + n_a(i) + \frac{c}{N} - \frac{1}{N} \sum_{l=0}^{\infty} h(l) \quad (2.23)$$

where

$$h_{df}(i) = \sum_{j=-\infty}^{\infty} h_d(i+jN), \quad (2.24)$$

and

$$n_a(i) = \frac{1}{MN} \sum_{k=1}^{MN} n(k) a(k-i). \quad (2.25)$$

The dc response of the system is

$$\sum_{l=0}^{\infty} h_d(l) = 0 \quad (2.26)$$

since the acoustic measurement system is AC coupled. We note that (2.24) may be considered as folding in the time domain. In order to further analyse the estimate we assume that (1.8) is valid, i.e.

$$\frac{1}{h_o^2} \int_0^{\infty} h^2(\tau) d\tau = 10^{-6} \frac{T_o}{T_R} \quad (2.27)$$

where

$$h_o^2 = \int_0^{\infty} h^2(\tau) d\tau \quad (2.28)$$

and thereby

$$\frac{1}{h_o^2} \sum_{k_o}^{\infty} h_d^2(k) \approx 10^{-6} \frac{k_o T_s}{T_R} \quad (2.29)$$

This means that by choosing

$$NT_s \geq T_R \quad (2.30)$$

these time domain folding errors become negligible i.e.

$$h_{df}(i) \approx h_d(i) \quad \text{for } i = 0, 1, \dots, N-1. \quad (2.31)$$

Furthermore we note that the noise signal $n_a(i)$ defined in (2.25) is a zero-mean periodic process. The variance of the process is upper bounded by

$$\sigma_{n_a}^2(i) \leq \sigma_{MAX}^2 = \frac{1}{MN} \max_v \{R_n(v)\} \quad (2.32)$$

where $R_n(v)$ is the spectral density of $n(k)$. This is shown in the Appendix and it is also shown that

$$\sigma_{MAX}^2 \approx \frac{1}{MN} \cdot \frac{f_s}{2B} \cdot \sigma_n^2 \quad (2.33)$$

if the sampling procedure gives negligible folding of the noise, the band-pass filter is rather sharp and the ripple in the pass-band small. For 1/1-octave filters we get using (2.15)

$$\sigma_{MAX}^2 \approx \frac{1}{MN} \cdot 2 \cdot \sqrt{2} \cdot \sigma_n^2. \quad (2.34)$$

The mean value of the variance over one period is defined by

$$\sigma_{MEAN}^2 = \frac{1}{N} \sum_{i=0}^{N-1} \sigma_{n_a}^2(i). \quad (2.35)$$

We find

$$\sigma_{MEAN}^2 = \frac{(N+1)}{N} \cdot \frac{1}{MN} [\sigma_n^2 + 2 \sum_{j=1}^{M-1} r_n(jN) (1 - \frac{j}{M})] - \frac{1}{N} E\{(\frac{1}{MN} \sum_{k=1}^{MN} n(k))^2\}. \quad (2.36)$$

We note that

$$\sigma_{MEAN}^2 \leq \frac{(N+1)}{N} \cdot \frac{1}{MN} \cdot \sigma_n^2 [1 + \frac{2}{\sigma_n^2} \sum_{j=1}^{M-1} |r_n(jN)| (1 - \frac{j}{M})]. \quad (2.37)$$

In the Appendix it is shown that

$$E\{(\frac{1}{MN} \sum_{k=1}^{MN} n(k))^2\} \leq \sigma_{MAX}^2. \quad (2.38)$$

This means that we also have a lower bound for σ_{MEAN}^2 given by

$$\sigma_{MEAN}^2 \geq \frac{(N+1)}{N} \cdot \frac{1}{MN} \cdot \sigma_n^2 [1 - \frac{2}{\sigma_n^2} \sum_{j=1}^{M-1} |r_n(jN)| (1 - \frac{j}{M})] - \frac{\sigma_{MAX}^2}{N}. \quad (2.39)$$

We note that for any fixed M there exists a large N such that

$$\sigma_{\text{MEAN}}^2 \approx \frac{\sigma_n^2}{MN} . \quad (2.40)$$

Let us rewrite (2.23) as

$$z_a(i) = \frac{(N+1)}{N} h_{df}(i) + n_a(i) + \frac{c}{N} \quad (2.41)$$

which shows that $z(i)$ consists of the folded pulse response $h_{df}(i)$ plus zero-mean noise and a dc error. We note that the dc error is suppressed by a factor $\frac{1}{N}$ and the variance of the noise by a factor proportional to $\frac{1}{MN}$. This means that $z_a(i)$ constitutes a sensible estimate of the pulse response $h_{df}(i)$. In situations where the dc error is extremely high one might instead prefer to estimate the pulse response by means of cross-correlating $y(k)$ and $b(k)$, defined in (2.20). That is

$$z_b(i) = \frac{1}{MN} \sum_{k=1}^{MN} y(k) b(k-i). \quad (2.42)$$

We find that this estimate

$$z_b(i) = z_a(i) - \frac{1}{N} \cdot \frac{1}{MN} \sum_{k=1}^{MN} y(k) = \frac{(N+1)}{N} h_{df}(i) + n_b(i) \quad (2.43)$$

eliminates the dc error since $n_b(i)$ is a zero-mean periodic process. The variance of $n_b(i)$ is upper bounded by

$$\sigma_{n_b}^2(i) = E\{n_b^2(i)\} \leq \sigma_{\text{MAX}}^2 . \quad (2.44)$$

Furthermore we have

$$E\left\{\frac{1}{N} \sum_{i=0}^{N-1} \sigma_{n_b}^2(i)\right\} \leq \sigma_{\text{MEAN}}^2 . \quad (2.45)$$

These formulas are derived with the same technique as used for $n_a(i)$. We may now define an input signal-to-noise ratio by

$$\text{SNR}_{\text{IN}} = \frac{1}{N\sigma_n^2} \sum_{k=1}^N y_a^2(kT_s) = \frac{(N+1)}{N\sigma_n^2} \sum_{\ell=0}^{N-1} h_{df}^2(\ell) \quad (2.46)$$

and an output signal-to-noise ratio

$$\text{SNR}_{\text{OUT}}(i) = \frac{(N+1)}{N} h_{\text{df}}^2(i) / \left(E \left\{ \frac{1}{N} \sum_{k=1}^N n_a^2(k) \right\} \right) \quad (2.47)$$

For large values of N , (2.40), we have

$$\text{SNR}_{\text{OUT}}(i) \approx MN \frac{h_{\text{df}}^2(i)}{\sigma_n^2} \quad (2.48)$$

We note that

$$\text{SNR}_{\text{IN}} \approx \text{SNR}_{\text{BP}} \quad (2.49)$$

and further that the gain in SNR is

$$G_{\text{SNR}}(i) = \frac{\text{SNR}_{\text{OUT}}(i)}{\text{SNR}_{\text{IN}}} \approx \frac{h_{\text{df}}^2(i)}{\frac{1}{MN} \sum_{k=0}^{N-1} h_{\text{df}}^2(k)}. \quad (2.50)$$

Our next task is to implement (2.22) or (2.42). This means designing a signal processing device. The block structure of such a device is shown in Fig. 2.2.

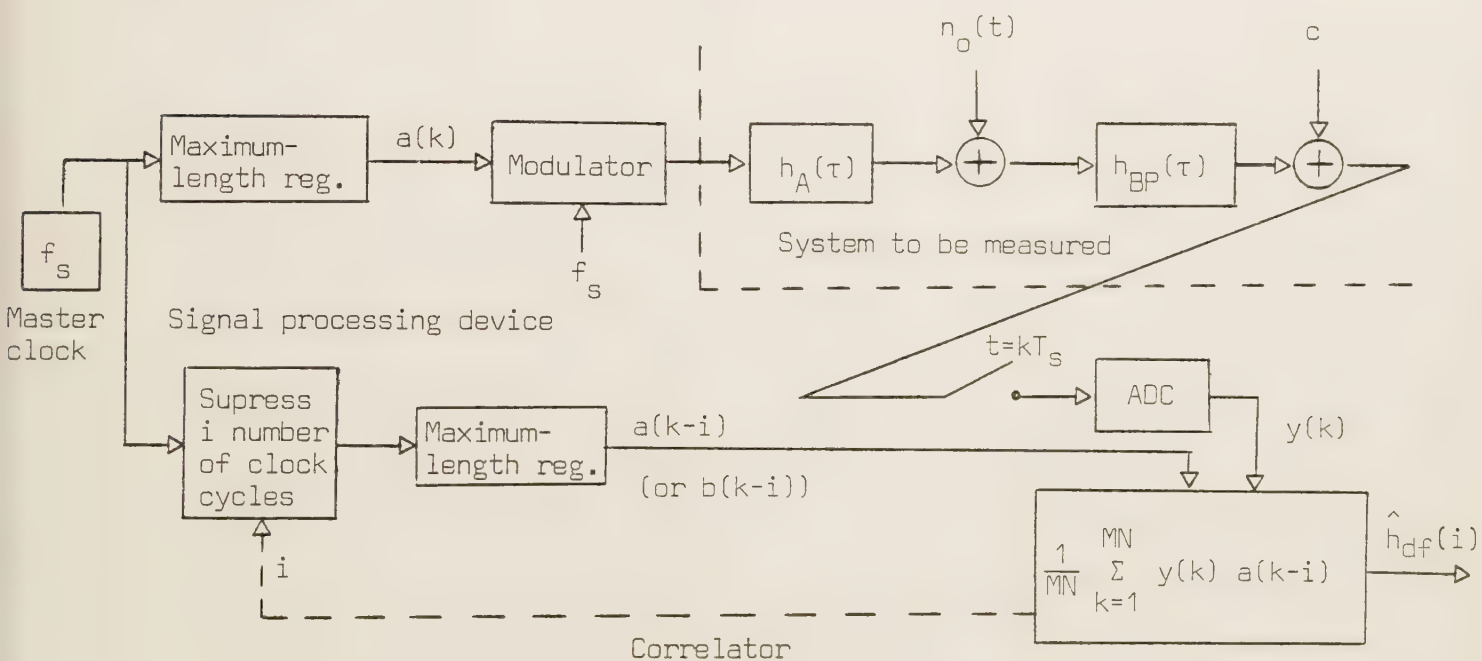


Figure 2.2. Blockstructure of the signal processing device.

The correlator may be realized in several ways. One way is indicated in Fig. 2.2 and another is to use an FFT algorithm. Here is, however, a problem since the period of the maximum-length sequence is $2^n - 1$ and the FFT algorithm can only handle periods of 2^m samples (n and m are integers). In literature one has suggested a few methods to avoid this problem. (It is of course also possible to use DFT but then to the price of a very long calculation time for large values of n .)

Fasbender and Günzel, [5], propose the use of different but coupled frequencies f_a and f_m such that

$$f_a = \frac{2^n}{2^n - 1} f_m . \quad (2.51)$$

Schroeder, [2], suggests extrapolation from $2^n - 1$ to 2^n samples in the period but gives no details. In this paper we will show that it is possible to ascertain the periodic correlation, $2^n - 1$, by means of combining the elements of the output vector from an FFT calculation of 2^{n+1} samples. Shortly we will present this method in detail and then calculate by how much the computational work increases.

We have two sequences $y(k)$ and $a(k)$. The period of $a(k)$ is $N = 2^n - 1$. We want to calculate one period of the sequence

$$z_a(i) = \frac{1}{MN} \sum_{k=1}^{MN} y(k) a(k-i) \quad (2.52)$$

We note that

$$z_a(i) = \frac{1}{N} \sum_{k=1}^N \bar{y}(k) a(k-i) \quad (2.53)$$

where

$$\bar{y}(k) = \frac{1}{M} \sum_{\ell=0}^{M-1} y(k+\ell N) . \quad (2.54)$$

We now define two new periodic sequences

$$y_1(k) = \begin{cases} \bar{y}(k) & k = 1, 2, \dots, N \\ 0 & k = N+1, \dots, 2(N+1) \end{cases} \quad (2.55)$$

and

$$a_1(k) = \begin{cases} a(k) & k = 1, 2, \dots, N \\ 0 & k = N+1, \dots, 2(N+1) . \end{cases} \quad (2.56)$$

The period of these sequences is 2^{n+1} . By means of the calculations for 2^{n+1} samples we get

$$A_1\left(\frac{\ell}{L}\right) = \sum_{k=1}^{2^{n+1}} a_1(k) e^{-j2\pi \frac{\ell}{L} k} \quad (2.1)$$

and

$$Y_1\left(\frac{\ell}{L}\right) = \sum_{k=1}^{2^{n+1}} y_1(k) e^{-j2\pi \frac{\ell}{L} k} \quad (2.2)$$

where

$$L = 2^{n+1}. \quad (2.3)$$

We use the relation

$$\sum_{k=1}^{2^{n+1}} y_1(k) a_1(k-i) \xleftrightarrow{\text{DFT}} Y_1\left(\frac{\ell}{L}\right) A_1^*\left(\frac{\ell}{L}\right) \quad (2.4)$$

where $*$ denotes complex conjugate. Note that $A_1^*\left(\frac{\ell}{L}\right)$ may be calculated in advance once and for all. We obtain, after inverse FFT, the periodic sequence

$$q(i) = \sum_{k=1}^{2^{n+1}} y_1(k) a_1(k-i) \quad (2.5)$$

and observe that

$$z_a(i) = \frac{1}{N} [q(i) + q(2^{n+1}+i)] \quad \text{for } i = 0, 1, \dots, N-1. \quad (2.6)$$

Let us illustrate this by means of a simple example where $N=6$ and $N=6$, i.e. $n=2$.

We wish to calculate $z(i)$ for $i = 0, 1, 2$. That is

$$\begin{cases} z_a(0) \cdot N = y(1) a(1) + y(2) a(2) + y(3) a(3) \\ z_a(1) \cdot N = y(1) a(3) + y(2) a(1) + y(3) a(2) \\ z_a(2) \cdot N = y(1) a(2) + y(2) a(3) + y(3) a(1) \end{cases} \quad (2.7)$$

We use $a_1(k)$, $y_1(k)$, (2.60) and find $q(i)$. Let us define the vectors

$$\underline{a}_1 = (a(1), a(2), a(3), 0, 0, 0, 0, 0) \quad (2.64)$$

and

$$\underline{y}_1 = (y(1), y(2), y(3), 0, 0, 0, 0, 0). \quad (2.65)$$

It is now easy to see that $q(i)$ is given by

$$\left\{ \begin{array}{lcl} q(0) & = & y(1) a(1) + y(2) a(2) + y(3) a(3) \\ q(1) & = & y(2) a(1) + y(3) a(2) \\ q(2) & = & y(3) a(1) \\ q(3) & = & 0 \\ q(4) & = & 0 \\ q(5) & = & 0 \\ q(6) & = & y(1) a(3) \\ q(7) & = & y(1) a(2) + y(2) a(3) \end{array} \right. \quad (2.66)$$

and that (2.62) gives us the $z(i)$ values.

The method presented in this paper is very simple to use. It needs however an increased calculation time. The calculation time for the FFT algorithm is proportional to $\ell \cdot 2^\ell$ where 2^ℓ is the number of samples. This means that the method approximately doubles the calculation time. Furthermore we also have to make N additions.

One question to be answered is: why have we divided the total number of samples into M number of groups, periods, each containing N number of samples? First we note that in acoustic applications the figure N has to be quite large, often much larger than 1000. This is due to the fact that acoustical systems have in general an extremely large bandwidth-time product. That means that when using FFT algorithms, specially on small computers, N has to assume the smallest possible value. The extra gain in signal-to-noise ratio, if necessary, is then found by increasing M . However if an on line working correlator is used there is no reason for having M greater than one since there is a small drawback in using M greater than one, e.g. the time domain folding (2.24).

3. ESTIMATION OF THE REVERBERATION DECAY FUNCTION

A. Introduction

We now use the estimate of the impulse response from chapter 2 to calculate an estimate of the reverberation decay function. In this chapter we will do the analysis in analog time. This means that we assume that the discrete time system in chapter 2 is reconstructed into an analog system. We assume that we have done an ideal or almost ideal reconstruction. The reconstructed estimate of the impulse response is

$$z(t) = h_f(t) + \eta(t) \quad (3.1)$$

where

$$h_f(kT_s) \simeq h_{df}(k) \simeq h_d(k) \simeq h(kT). \quad (3.2)$$

The signal $\eta(t)$ is a zero mean random process assumed to be in wide sense stationary. This assumption is valid if one introduces in the exciting signal a random time shift equally distributed over one period. This is a commonly used technique in communication theory and is equivalent to the one used in chapter 2 where we made calculations over one period of the noise signal, (2.35). The variance of the process $\eta(t)$ is bounded by

$$\sigma_\eta^2 \leq \sigma_{\text{MAX}}^2 \simeq \sigma_n^2 \cdot \frac{1}{MN} \cdot \frac{f_s}{2B} \quad (3.3)$$

and for any fixed M there exists a large N such that

$$\sigma_\eta^2 \simeq \frac{\sigma_n^2}{MN}. \quad (3.4)$$

We recall from chapter 1 that the reverberation decay function is defined by

$$r(t) = \frac{1}{h_0^2} \int_0^\infty h^2(\tau) d\tau \quad (1.1)$$

where

$$h_0^2 = \int_0^\infty h^2(\tau) d\tau. \quad (2.28)$$

We also have

$$h_o^2 \approx T_s \cdot \sum_{\ell=0}^{N-1} h_{df}^2(\ell). \quad (3.5)$$

We assume that (3.2) and (3.5) are valid approximations.

B. Analysis of the estimated reverberation decay function

We are now ready to calculate an estimate of $r(t)$ in the interval $0 \leq t < T_o$. This estimate is given by

$$\hat{r}(t) = \frac{1}{h_o^2} \int_t^{T_o} z^2(\tau) d\tau = \frac{1}{h_o^2} \int_t^{T_o} (h(\tau) + \eta(\tau))^2 d\tau. \quad (3.6)$$

The mean value of $\hat{r}(t)$ is

$$m_r(t) = E\{\hat{r}(t)\} = \frac{1}{h_o^2} \left[\int_t^{T_o} h^2(\tau) d\tau + (T_o - t) \sigma_\eta^2 \right]. \quad (3.7)$$

The relative mean error is defined by

$$\epsilon(t) = \frac{m_r(t) - r(t)}{r(t)}. \quad (3.8)$$

In order to analyse the estimate we assume that $r(t)$ has the form

$$r(t) = e^{-t \frac{6 \ln 10}{T_R}} \quad (1.8)$$

Thereby we find

$$\epsilon(t) = \frac{(T_o - t) \sigma_\eta^2}{h_o^2} e^{t \frac{6 \ln 10}{T_R}} - e^{-(T_o - t) \frac{6 \ln 10}{T_R}} \quad (3.9)$$

and note that we have two types of errors, noise power integration and truncation. We choose to deal with them separately since this will give conservative error estimates and as will be shown, a handy method. The error tolerance of the truncation is chosen to be

$$e^{-(T_o - t) \frac{6 \ln 10}{T_R}} \leq e^{-1} \quad (3.10)$$

and of the noise power integration

$$\frac{(T_o - t) \sigma_n^2}{h_o^2} e^{t \frac{6 \ln 10}{T_R}} \leq e^{-1} . \quad (3.11)$$

The formula (3.10) specifies the upper bound T_M as

$$t \leq T_M = T_o - \frac{T_R}{6 \ln 10} . \quad (3.12)$$

The function on the left side of (3.11) has a maximum value for $t = T_M$ equal to

$$g_M = g(T_M) = \frac{\sigma_n^2}{h_o^2} \cdot \frac{T_R}{6 \ln 10} e^{(T_o \cdot \frac{6 \ln 10}{T_R} - 1)} . \quad (3.13)$$

The tolerance (3.11) then gives

$$T_o \leq T_{oM} = \frac{T_R}{6 \ln 10} \cdot \ln \left(\frac{h_o^2}{\sigma_n^2} \cdot \frac{6 \ln 10}{T_R} \right) . \quad (3.14)$$

One wants the estimate $\hat{r}(t)$ to be valid over the largest possible interval. This means that we choose $T_o = T_{oM}$ and consider $\hat{r}(t)$ given in the interval

$$0 \leq t \leq T_{oM} - \frac{T_R}{6 \ln 10} . \quad (3.15)$$

We will also calculate the variance of the estimate $\hat{r}(t)$. We do this by assuming $\eta(t)$ to be Gaussian distributed. One finds

$$\begin{aligned} \sigma_{\hat{r}}^2(t) &= E\{(\hat{r}(t) - m_r(t))^2\} = \\ &= \frac{4}{h_o^4} \int_t^{T_o} \int_t^{T_o} h(\tau_1) h(\tau_2) r_{\eta}(\tau_1 - \tau_2) d\tau_1 d\tau_2 + \frac{2}{h_o^4} \int_t^{T_o} \int_t^{T_o} r_{\eta}^2(\tau_1 - \tau_2) d\tau_1 d\tau_2 \end{aligned} \quad (3.16)$$

where $r_{\eta}(\tau)$ is the covariance function of the noise $\eta(t)$. The variance is limited i.e.

$$\sigma_{\hat{r}}^2 \leq \sigma_{\max}^2(t) \quad (3.17)$$

where

$$\sigma_{\max}^2 \approx \frac{2\sigma_{\eta}^2}{h_o^4 B} \int_t^{T_o} h^2(\tau) d\tau + \frac{(T_o - t)}{h_o^4 B} \sigma_{\eta}^4. \quad (3.18)$$

The relative standard deviation is then also limited i.e.

$$\sigma_{\varepsilon}(t) = \frac{\hat{\sigma}_r(t)}{r(t)} \leq \frac{\sigma_{\max}(t)}{r(t)} = \sigma_{\varepsilon\max}(t) \quad (3.19)$$

where

$$\sigma_{\varepsilon\max}(t) = e^{t \frac{6\ln 10}{T_R}} \cdot \frac{\sigma_{\eta}}{\sqrt{B}} \left(\frac{2}{h_o^2} (e^{-t \frac{6\ln 10}{T_R}} - e^{-T_o \frac{6\ln 10}{T_R}}) + (T_o - t) \frac{\sigma_{\eta}^2}{h_o^4} \right)^{1/2}. \quad (3.20)$$

This function, $\sigma_{\varepsilon\max}(t)$, is an increasing function in the interval $0 \leq t < T_o - T_R/6\ln 10$. This means that in this interval we have

$$\sigma_{\varepsilon\max}(t) \leq \sigma_{\varepsilon\max}(T_o - \frac{T_R}{6\ln 10}) \leq \sqrt{\frac{2e-1}{e^2 B T_R}} < \frac{1}{\sqrt{B T_R}} \quad (3.21)$$

where

$$T_R = \frac{T_R}{6\ln 10}. \quad (3.22)$$

We note that this variance error often becomes negligible. However for low frequency bands and short reverberation times this error has to be considered and possibly decreased by e.g. repeated measurements.

One generally wants to determine a reverberation decay function estimate defined for $0 \leq t \leq T_R \cdot 2/3$. This implies

$$\frac{T_R \cdot 2}{3} \leq T_o - \frac{T_R}{6\ln 10} = \frac{T_R}{6\ln 10} \left(\ln\left(\frac{h_o^2}{\sigma_{\eta}^2} \cdot \frac{6\ln 10}{T_R}\right) - 1 \right) \quad (3.23)$$

and a period time

$$T_P = N T_s \geq \left(\frac{2}{3} + \frac{1}{6\ln 10} \right) T_R. \quad (3.24)$$

We find

$$\text{SNR}_{\text{IN}} \geq 10^4 \frac{e}{6 \ln 10} \cdot \frac{f_s}{2B} \cdot \frac{T_R}{T_{\text{EST}}} \quad (3.25)$$

where SNR_{IN} is defined in chapter 2 and T_{EST} is the total estimation time

$$T_{\text{EST}} = M N T_s = M T_P. \quad (3.26)$$

If we use a 1/1-octave filter and $M=1$, $T_P = T_R$ we get

$$\text{SNR}_{\text{IN}} \geq 10^4 \frac{e\sqrt{2}}{3 \ln 10} \approx 0.6 \cdot 10^4 \approx 38 \text{ dB}. \quad (3.27)$$

Conversely, if SNR_{IN} is known and a rough estimate of T_R given, (3.25) may be used to calculate the required total estimation time.

It is possible to define a time dependent output signal-to-noise ratio by

$$\text{SNR}_{\text{OUT}}(T) = \frac{\frac{T_0}{T} \int_0^T h^2(\tau) d\tau}{(T_0 - T) \sigma_n^2} \quad \text{for } 0 \leq T \leq T_M. \quad (3.28)$$

This gives

$$\text{SNR}_{\text{OUT}}(T) \geq \frac{h_0^2 10^{-6} \frac{T}{T_R}}{(T_0 - T) \sigma_n^2 \cdot \frac{f_s}{2B} \cdot \frac{1}{MN}} = 10^{-6} \frac{T}{T_R} \cdot \frac{T_P}{(T_0 - T)} \cdot M \cdot \frac{2B}{f_s} \text{SNR}_{\text{IN}} \quad (3.29)$$

We claimed in chapter 1 that the method of using maximum-length binary sequences gives a higher signal-to-noise ratio than the direct impulse method. This is true since with the direct method, using a loudspeaker as exciting device, it is difficult to reach a high input signal-to-noise ratio. The loudspeaker and the amplifier for the loudspeaker can only correctly handle limited amplitude signals. When using maximum-length sequences the exciting signal is

$\sum_{k=-\infty}^{\infty} s(t - kT_s) \cdot a(k)$. This means that the input signal energy of one period is approximately $N \cdot \int_{-T_s/2}^{T_s/2} s^2(\tau) d\tau = N \cdot E_s$

since the pulses $s(t - kT)$ are non-overlapping or almost non-overlapping.

When using the direct method the exciting signal is $s(t)$. This means that the input signal energy of one period is equal to E_s . This gives an N times lower input signal-to-noise ratio. The analytical basis is otherwise iden-

tical. This means that the output signal-to-noise ratio of the direct method is

$$\text{SNR}_{\text{OUT}}^{\text{D}}(T) = M_{\text{D}} \frac{\int_0^{T_{\text{OD}}} h^2(\tau) d\tau}{(T_{\text{OD}} - T) \sigma_n^2} \quad \text{for} \quad 0 \leq T \leq T_{\text{MD}} \quad (3.30)$$

where M_{D} , T_{OD} and T_{MD} are defined analogical to M , T_{O} and T_{M} . This gives

$$\text{SNR}_{\text{OUT}}^{\text{D}}(T) \geq 10^{-6} \frac{T}{T_{\text{R}}} \frac{T_{\text{P}}}{(T_{\text{OD}} - T)} \cdot M_{\text{D}} \cdot \text{SNR}_{\text{IN}}. \quad (3.31)$$

We conclude that the gain in signal-to-noise ratio by using maximum-length sequences instead of impulses is proportional to N .

C. Practical considerations

At this stage we will use the formulas from section B in a practical measurement. The impulse response is estimated by $z(t)$ in the interval $0 \leq t \leq T_{\text{P}}$.

At first we have to find $T_{\text{O}} = T_{\text{M}}$ but T_{M} is given by (3.14) and requires knowledge of h_{O}^2 , σ_n^2 and T_{R} . We therefore have to use rough estimates of these parameters. A rough estimate of σ_n^2 is

$$\hat{\sigma}_n^2 = \frac{1}{\Delta T} \int_{T_{\text{P}} - \Delta T}^{T_{\text{P}}} z^2(t) dt \quad (3.32)$$

where $T_{\text{P}} - \Delta T \leq t \leq T_{\text{P}}$ ought to be an interval in which the signal-to-noise ratio is less than 1. The interval length, ΔT , shall if possible be chosen such that $B\Delta T \gg 1$.

A rough estimate of h_{O}^2 is

$$\hat{h}_{\text{O}}^2 = \int_0^{T_{\text{P}}} z^2(t) dt \quad (3.33)$$

and finally a rough estimate of T_{R} is

$$\hat{T}_{\text{R}} = 6(t_{-15} - t_{-5}) \quad (3.34)$$

where

$$\int_{t_{-5}}^{T_P} z^2(t) dt = \hat{h}_0^2 \cdot 10^{-5/10} \quad (3.35)$$

and

$$\int_{t_{-35}}^{T_P} z^2(t) dt = \hat{h}_0^2 \cdot 10^{-15/10} \quad (3.36)$$

Having found these rough estimates we use them in (3.14) and find an estimate of T_0 and thereby also an estimate of the reverberation decay function. The rough estimate of σ_η^2 is often bad. The other estimates, $\hat{h}_0^2$ and $\hat{T}_R$, are more seldom bad. The former is in general the case if one has received high signal-to-noise ratio. Since then (3.32) estimates signal power and not noise power.

However if the input signal-to-noise ratio has been measured separately a better estimate of σ_η^2 will then be

$$\hat{\sigma}_\eta^2 = \frac{\hat{h}_0^2}{M T_P} \cdot \frac{1}{\text{SNR}_{BP}}$$

where SNR_{BP} is defined in chapter 2, (2.12).

It remains to study the errors caused by these rough estimates. This is difficult to do since one major error arises when (1.8) is not valid. The effects of these rough estimates are however illustrated by the measurements presented in the following chapter.

4. A MEASUREMENT AND SOME CONCLUDING REMARKS

A. Measurement

The problem of obtaining large signal-to-noise ratios is often pronounced in the lower frequency bands. We have therefore made a few measurements at low frequencies. We here present a measurement of the impulse response and the corresponding reverberation decay function for the 125 Hz 1/1-octave band. The sampling frequency is 500 Hz and we have used a maximum-length sequence with the period 4095 samples. This gives the period time $T_P = 8.19$ seconds. The measurement is made over one period i.e. $M=1$. In Fig. 4.1 the estimated impulse response is plotted and in Fig. 4.2 the corresponding reverberation decay function. The rough estimates presented in chapter 3, section C, are used assuming an unknown input signal-to-noise ratio and using $\Delta T = 0.6$ seconds. The method of using rough estimates is illustrated in Fig. 4.2. We have namely assumed $z(t)$ known for a time interval $0 \leq t \leq T_Z$, shorter than the one originally given. Applying the rough estimates then means replacing T_P by T_Z in (3.32)-(3.36). This gave different rough estimates of T_O , h_O^2 , σ_n^2 and T_R . The result is presented in Table 4.1 for $T_Z = 2, 4, 6$ and 8 seconds. The values of $\hat{h}_O^2$ and $\hat{\sigma}_n^2$ are given in units $\times$ units $\times$ seconds and units $\times$ units respectively where the unit is arbitrary.

T_Z	$\hat{T}_O$	$\hat{h}_O^2$	$\hat{\sigma}_n^2$	$\hat{T}_R$
8s	2.690s	$6.19 \cdot 10^3$	0.66	3.300s
6s	2.678s	$6.19 \cdot 10^3$	0.70	3.300s
4s	2.446s	$6.19 \cdot 10^3$	1.50	3.300s
2s	1.254s	$6.17 \cdot 10^3$	264.16	3.228s

Table 4.1. The rough estimates for $z(t)$ known for different time intervals $0 \leq t \leq T_Z$.

We note from Table 4.1 that $\hat{h}_O^2$ and $\hat{T}_R$ are almost constant for the different values of T_Z . However $\hat{\sigma}_n^2$ is not constant for all the different values of T_Z . This is due to the fact that for small T_Z , $\hat{\sigma}_n^2$ estimates not the noise power but the signal power. This means that the reverberation decay function estimate is valid for an unnecessarily short time interval. The deviation between the estimates in Figure 4.2 is small and within the error tolerance, i.e. (3.10) and (3.11).

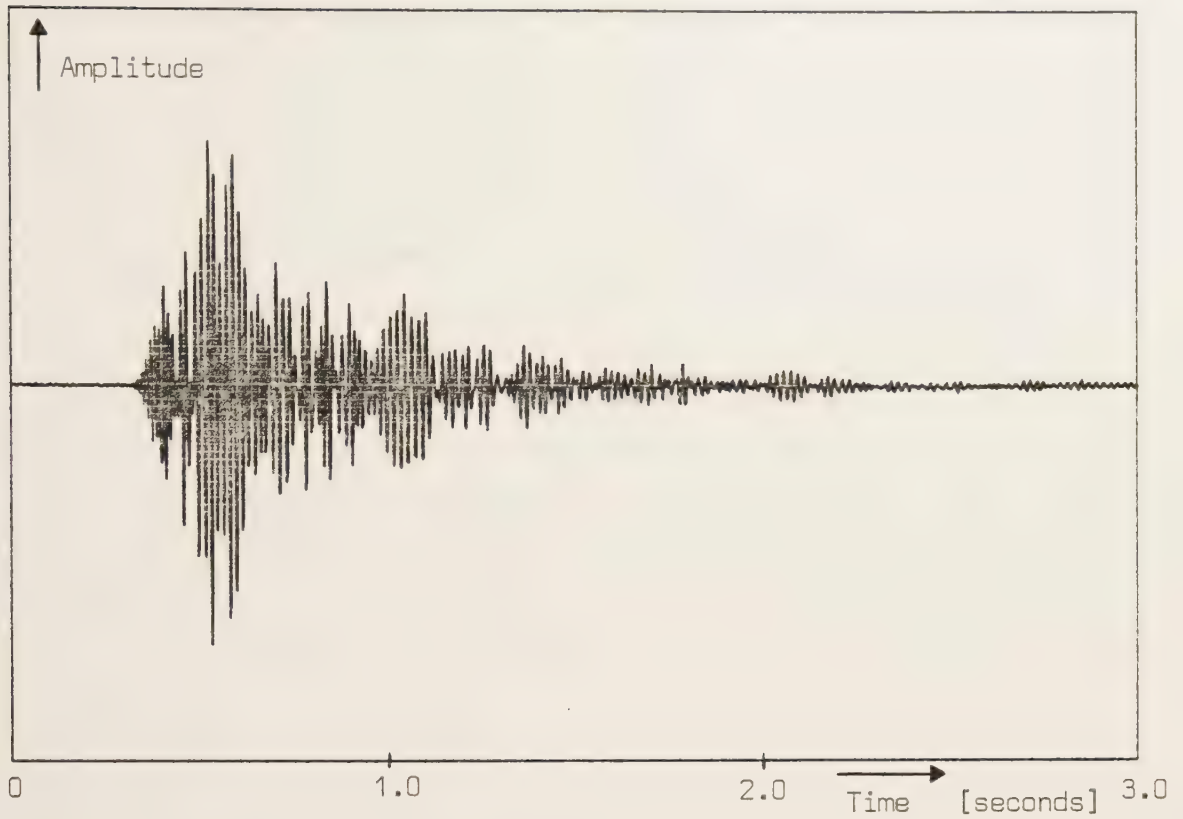


Figure 4.1. Estimated impulse response of the 125 Hz 1/1-octave frequency band.

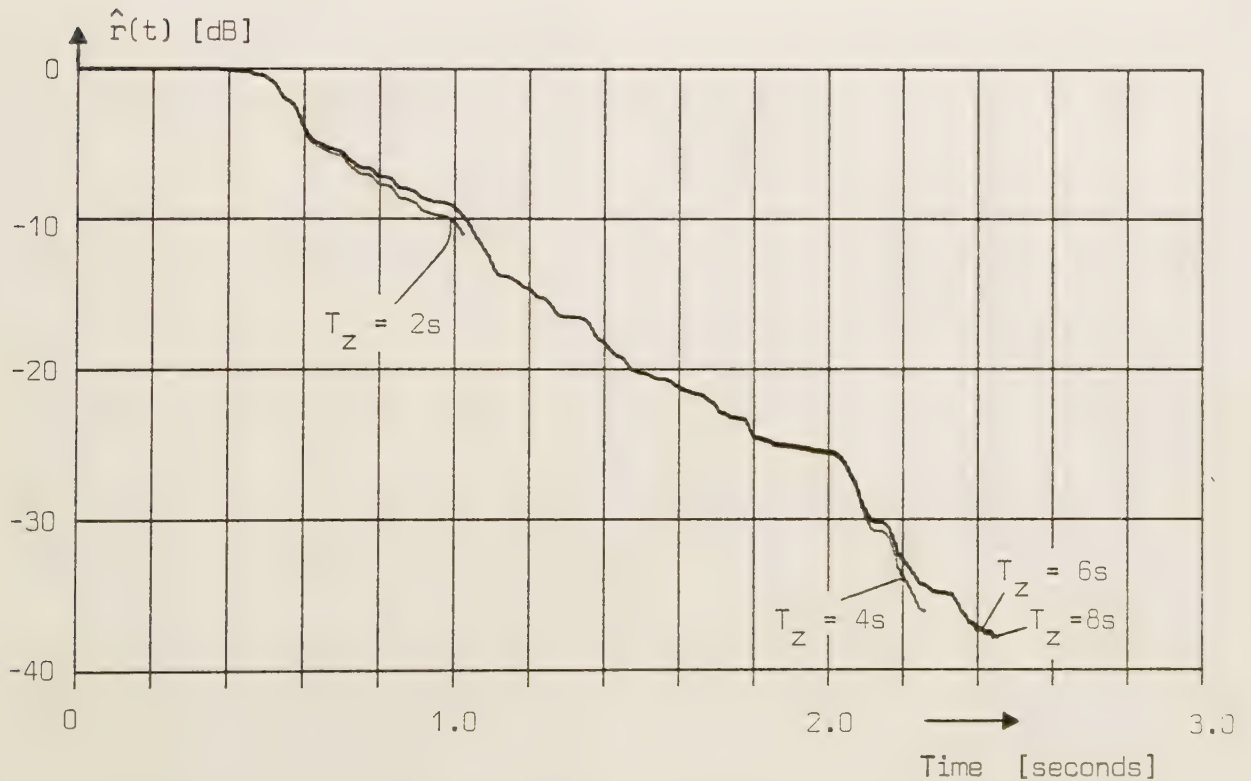


Figure 4.2. The estimated reverberation decay function of the 125 Hz 1/1-octave frequency band for $T_Z = 2, 4, 6$ and 8 seconds.

The input signal-to-noise ratio of this experiment has not been measured. However by using the estimates $\hat{h}_0^2$ and $\hat{\sigma}_n^2$ for $T_Z = 8$ we may represent such a measurement and thereby also illustrate the effects of using it to find the rough estimates. This gives $\hat{T}_0 \approx 2.6s$ for $T_Z = 2, 4, 6$ or $8s$ and for $T_Z = 2$ we have to use $\hat{T}_0 = 2s$. Using this "represented SNR_{IN} " we get estimates of the reverberation decay function shown in Fig. 4.3. We note that this gives estimates valid over a larger time interval. However we also observe a deviation of about 4 dB for $T_Z = 2$. This deviation is not within the error tolerances. The large deviation is due to the fact that (1.8) is only approximately valid. In the interval $1.8s \leq t \leq 2s$ the decay function decreases less than what is expected from the formula (1.8). This gives larger errors than expected and indicates that the deviation of $r(t)$ from (1.8) also needs to be in some way included in the calculations. This requires however statistics of how $r(t)$ deviates from (1.8) and is outside the scope of this paper.

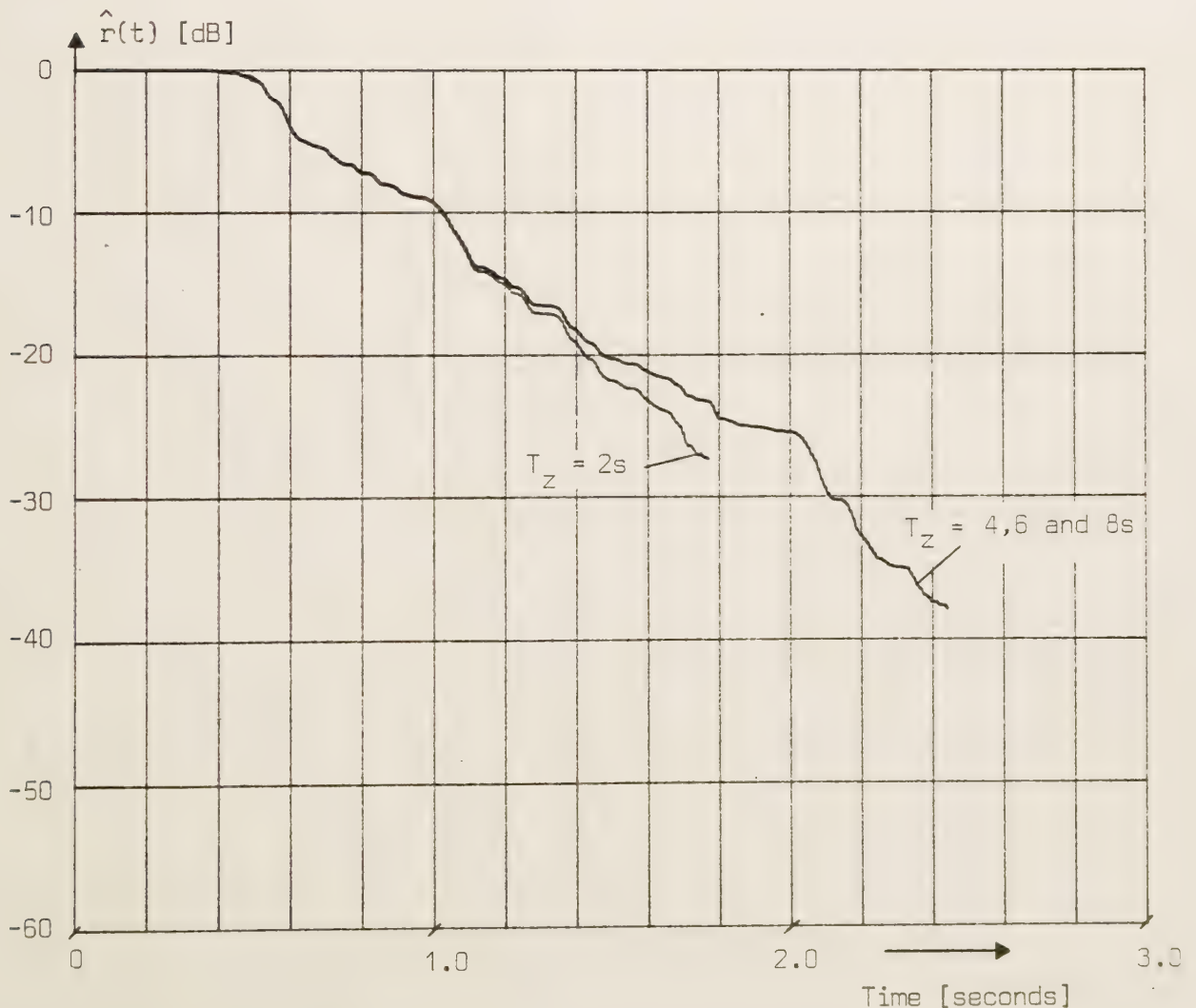


Figure 4.3. The estimated reverberation decay function of the 125 Hz 1/1-octave frequency band for $T_Z = 2, 4, 6$ and 8 seconds using a simulated SNR

B. Some concluding remarks

We have in this example and in the figures 4.2 and 4.3 seen that the method worked well and gave relatively well defined reverberation decay functions of high dynamic range. We have noted that it might be an essential advantage if the input signal-to-noise ratio is separately measured. The deviations between different estimates, made for the same acoustic system, were within the tolerance bounds in Fig. 4.2. In Fig. 4.3 they were outside the tolerance bounds due to the fact that the reverberation decay function doesn't decay exponentially.

The method suggested in this paper is very simple and thereby practical. We therefore hope that this method will be frequently used in the future.

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APPENDIX

We start by giving and proving a lemma.

Lemma

Let $n(k)$ be an in wide sense stationary random process with limited spectral density $R_n(v)$ and a series, $\{c(k)\}_{k=-\infty}^{\infty}$, such that

$$\sum_{k=-\infty}^{\infty} c^2(k) = C < \infty. \quad (A1)$$

Then

$$E\left\{\left(\sum_{k=-\infty}^{\infty} c(k-i) n(k)\right)^2\right\} \leq \sigma_M^2 C \quad (A2)$$

where

$$\sigma_M^2 = \max_v \{R_n(v)\}. \quad (A3)$$

Proof

Define a zero-mean in wide sense stationary process $n_c(k)$ such that

$$E\{n(k) n_c(l)\} = 0 \quad \forall(k, l) \quad (A4)$$

and

$$n_w(k) = n(k) + n_c(k) \quad (A5)$$

becomes a white process with a variance σ_M^2 as small as possible. The spectral density of $n_c(k)$ is

$$R_{n_c}(v) = \sigma_M^2 - R(v). \quad (A6)$$

Formula (A3) then assures that the process $n_c(t)$ exists. Finally (A2) is received by noting

$$\begin{aligned}
E\left\{\left(\sum_{k=-\infty}^{\infty} c(k-i)n(k)\right)^2\right\} &\leq E\left\{\left(\sum_{k=-\infty}^{\infty} c(k-i)n(k)\right)^2\right\} + E\left\{\left(\sum_{k=-\infty}^{\infty} c(k-i)n_c(k)\right)^2\right\} = \\
&= E\left\{\left(\sum_{k=-\infty}^{\infty} c(k-i)n_w(k)\right)^2\right\} = \sigma_M^2 C
\end{aligned} \tag{A7}$$

and that ends the proof.

We now prove (2.32), i.e.

$$E\{n_a^2(i)\} \leq \frac{1}{MN} \cdot \sigma_M^2 \tag{A8}$$

by applying the lemma. The process $n_a(i)$ is defined in (2.25), i.e.

$$n_a(i) = \frac{1}{MN} \sum_{k=1}^{MN} a(k-i) n(k). \tag{A9}$$

We find

$$E\{n_a^2(i)\} \leq \frac{\sigma_M^2}{(MN)^2} \sum_{k=1}^{MN} a^2(k-i) = \frac{1}{MN} \sigma_M^2. \tag{A10}$$

Using the same technique it is obvious that

$$E\left\{\left(\frac{1}{MN} \sum_{k=1}^{MN} n(k)\right)^2\right\} \leq \frac{1}{MN} \sigma_M^2. \tag{A11}$$

We will now estimate σ_M^2 . The assumptions made in chapter 2 give

$$R(v) \approx f_s R_o |H_{BP}(vf_s)|^2 \tag{A12}$$

and

$$\sigma_M^2 \approx f_s \cdot R_o \tag{A13}$$

From chapter 2 we also know that

$$\sigma_n^2 = R_o \cdot 2B \tag{A14}$$

where B is the bandwidth of the filter. This gives

$$\sigma_M^2 \approx \sigma_n^2 \cdot \frac{f_s}{2B}. \tag{A15}$$

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TR-146	Sundberg Carl-Erik	Autocorrelation and crosscorrelation properties of sequences generated from cyclic error-correcting codes.
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